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Optimal Near-Minimum-Time Control

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Introduction

THE solutions to many minimum-time optimal control problems are bang-bang controls. These controls are characterized by the controls switching instantaneously between saturation levels during the maneuver.[1,2] Instantaneous switching of the controls, however, is difficult to implement and can excite flexible-body modes,[3] and so researchers over the years have developed and implemented several ad hoc shaping techniques to smooth the formal minimum-time solutions.[3–5] These solutions have come to be known as near-minimum-time control solutions.

In this Note, we introduce a new performance index that gives rise to optimally smooth near-minimum-time controls. Our performance measure penalizes a weighted combination of elapsed time and the first derivative of the controls. The control derivatives are penalized to smooth the control profiles and thereby discourage rapid switching between saturation levels. This is balanced with a constant parameter that weights the relative importance of elapsed time.

Problem Formulation

Our focus is on systems described by a set of ordinary differential equations with bounded controls. That is, $\dot{x}_i = f_i(x, u)$, where $-u_m^* \le u_m \le u_m^*$. We require that f_i is continuous and has continuous partial derivatives and that the admissible control u_m is continuous and has a first derivative that is piecewise continuous with no explicit bounds. We begin by demoting the controls to additional state variables, thus adding another layer of dynamics to the original system, and selecting the first derivative of each control as the new control variables. Hereafter we will refer to the new control variables as pseudocontrols. Thus, we have

$$\dot{x}_i = f_i(x, u), \qquad -u_m^* \le u_m \le u_m^* \tag{1}$$

$$\dot{u}_m = w_m \tag{2}$$

Our performance measure is given by

$$J = \frac{1}{2}\int_{t_0}^{t_f} [p + w_m w_m]\,\mathrm{d}t \tag{3}$$

The penalty on the pseudocontrols discourages rapid switching between saturation levels, whereas the constant parameter p weights the relative importance of elapsed time. It is clear that, as $p \to \infty$, the index becomes proportional to the formal minimum-time index.

We seek to minimize J in Eq. (3) over the admissible set of pseudocontrols, which are piecewise continuous functions with no explicit bounds, that map the interval $[t_0, t_f]$ into the space R^m, subject to the constraints given by Eqs. (1) and (2), along with specified boundary conditions involving the states x_i and u_m.

By demoting the true controls to additional state variables, our problem is now one that involves first-order state variable inequality constraints. Optimal control of systems possessing state variable inequality constraints has been researched for many years, and the necessary conditions are now well understood.[1,6,7] How the conditions for optimal control are manifest in the types of systems that we consider in this Note is discussed.

It is obvious that, over finite intervals when a constraint is active, the corresponding pseudocontrol must equal zero. That is,

$$\text{if} \quad |u_m| = u_m^*, \quad \text{then} \quad w_m = 0 \tag{4}$$

Also, Pontryagin's principle states that the Hamiltonian must be minimized over the set of all possible pseudocontrols, and this leads to the Kuhn–Tucker conditions on the Valentine multipliers μ_m (the Valentine multipliers enforce the differentiated form of the inequality constraints)

$$\begin{aligned} &\text{if} \quad |u_m| < u_m^*, \quad \text{then} \quad \mu_m = 0 \\ &\text{if} \quad |u_m| = u_m^*, \quad \text{then} \quad \mu_m \ge 0 \end{aligned} \tag{5}$$

The state and adjoint differential equations are given by

$$\dot{x}_i = f_i(x, u) \tag{6}$$

$$\dot{u}_m = w_m \tag{7}$$

$$\dot{v}_j = -v_i \frac{\partial f_i}{\partial x_j} \tag{8}$$

$$\dot{\rho}_m = -v_i \frac{\partial f_i}{\partial u_m} - \mu_l W_{lm} \tag{9}$$

where W_{lm} is a diagonal matrix with the elements w along the diagonal. Note that Eqs. (4) and (5) imply that the terms $\mu_l W_{lm}$ appearing in Eqs. (9) are always zero because either μ_m or w_m vanishes.

The optimality conditions follow from differentiating the Hamiltonian function with respect to the pseudocontrols and setting the results equal to zero. Using the optimality conditions and Eq. (5), we find: if $|u_m| < u_m^*$, then $w_m = -\rho_m$; and if $|u_m| = u_m^*$, then $\mu_m = -\rho_l U_{lm}^{-1}$.

Additional conditions include the transversality conditions on the state and adjoint variables, the Legendre–Clebsch condition (which is trivially satisfied for our problem), and the monotonicity of the Valentine multipliers, when the constraints are active. Indeed, the Valentine multipliers are nonincreasing functions of time when the constraints are active.[6] Furthermore, because the final time is free and the system is not an explicit function of time, the Hamiltonian function equals zero throughout.

In addition to the preceding statements, certain tangency conditions must be satisfied when entering and leaving the constraint boundaries. Recall that, when equality of a constraint is achieved, it is true that $|u_m| = u_m^*$ and $w_m = 0$. These equations represent interior boundary conditions that must be met whenever u_m saturates. Thus, instead of the usual two-point boundary-value problem, the necessary conditions define a multipoint boundary-value problem.

One other consequence of the inequality relations is that the costates ρ_m associated with the true control variables will be

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discontinuous when entering the constraint arcs, whereas all other variables and the Hamiltonian will be continuous.[1] Finally, note that, as the time spent on an active constraint boundary approaches zero, it is apparent that the limiting condition is a touchpoint wherein the variables u_m are allowed to kiss instantaneously and bounce off the constraints.

Numerical Solution Procedure

The preceding set of equations defining the multipoint boundary-value problem, Eqs. (4–9), is ideally suited for a multiple-shooting method of numerical solution.[8] This numerical method is fully explained in Refs. 2 and 8.

In our problem, the times when the true control variables u_m saturate and the final maneuver time are unknown. When using a shooting method to numerically solve a problem that contains unknown times, it is helpful to perform a change of time variables to convert each unknown time interval to a fixed time interval. For our multipoint boundary-value problem we utilize a set of nonlinear mappings. To illustrate, suppose that a control saturates at some unknown time t_1, and further suppose that the final time t_f is free. The nonlinear mappings that we consider have the form: if $t_0 \le t \le t_1$ then $t = t_0 + 3\kappa_1^2\tau^2 - 2\kappa_1^2\tau^3$, where $\tau \in [0, 1]$; and if $t_1 \le t \le t_f$ then $t = t_1 + 3\kappa_f^2\tau^2 - 2\kappa_f^2\tau^3$, where $\tau \in [0, 1]$. Here $\kappa_1^2 = t_1 - t_0$ and $\kappa_f^2 = t_f - t_1$ are unknown constants. Note that this change of variables from t to τ slightly modifies the performance index and Hamiltonian. The end result is that κ_1 and κ_f are treated as new state variables, which in turn introduce new corresponding costate variables. The overriding benefit is that variations in κ_1 and κ_f reflect variations in t_1 and t_f, and this is how these unknown times participate in the multiple-shooting method solution.

One downside of using the multiple shooting method is that one must first hypothesize the switching structure of the true controls u_m. Insight into the switching structure may be obtained by letting the parameter p be small enough such that none of the true control variables saturates and then increasing p and enforcing the state variable inequality constraints as they are encountered (this homotopy-inspired embedding approach is used subsequently in example 2). As p is increased, however, one begins to encounter touchpoints, and unfortunately, the nonlinear mappings discussed earlier can lead to numerical difficulties. To help prevent numerical difficulties, it may be necessary to increase p to step over the touchpoints to the situation where the constraints are active over finite intervals.

Example 1

Our first example is the classic double integrator. The equations for the system are

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \qquad \dot{u} = w$$

where $|u| \le 1$. We consider a rest-to-rest maneuver with the initial and final conditions on all of the state variables equal to zero, except that $x_1(t_0) = \pi/2$.

The analytical solution to the formal minimum-time problem for this system is well known and is shown by the solid line in Fig. 1, where the final time and switch time are given by $t_f = \sqrt{(2\pi)}$ and $t_s = t_f/2$. This analytical solution is helpful because it provides a limiting case baseline to compare the optimal near-minimum-time solution.

Optimal near-minimum-time solutions for two values of the parameter p are also shown in Fig. 1. We see that an increase in p corresponds to a decrease in final time and to a more rapid transition between saturation levels. We also see that the control transitions smoothly on entering and leaving the constraint boundaries. One unexpected result is that the slope of the control in transitioning to and from the constraint boundaries during the midportion of the maneuver is more shallow than at the beginning and end of the maneuver.

Example 2

Our next example is a near-minimum-time spacecraft reorientation problem. Variants of this problem have been studied in the recent past.[9]

We consider a prolate rigid body with the axis of symmetry along the third coordinate axis, where the inertias of the body are taken as

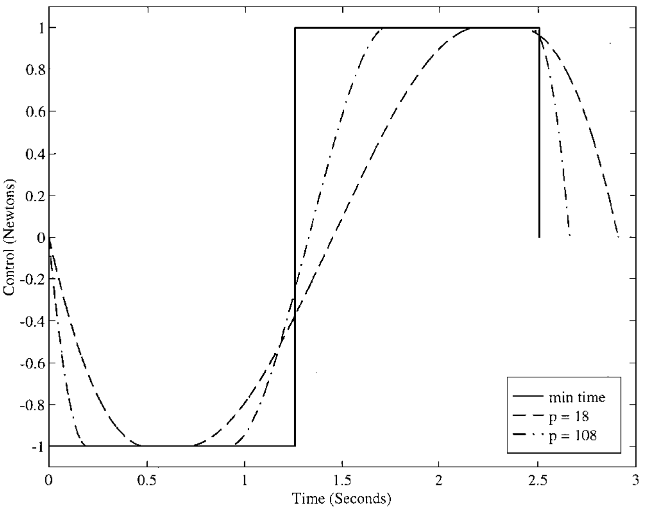

Fig. 1 Control profiles for example 1.

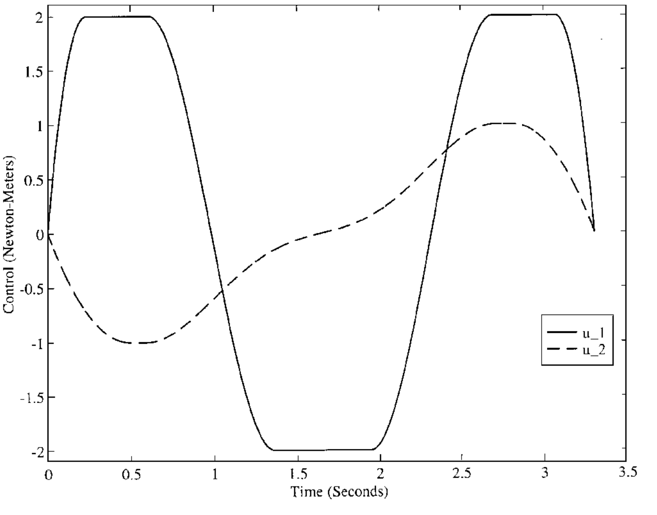

Fig. 2 Control profiles for example 2, p = 335.

$I_1 = I_2 = 1$ and $I_3 = \frac{1}{2}$. Furthermore, we consider bounded control about only two of the principal axes. The spacecraft is then governed by the rotational dynamic equations

$$\dot{\omega}_1 = \omega_2\omega_3/2 + u_1, \qquad |u_1| \le 2$$

$$\dot{\omega}_2 = -\omega_1\omega_3/2$$

$$\dot{\omega}_3 = 2u_2, \qquad |u_2| \le 1$$

and kinematic differential equations involving the modified Rodriguez parameters. We also have the differential equations defining the pseudocontrols $\dot{u}_1 = w_1$ and $\dot{u}_2 = w_2$. We consider a 180-deg, rest-to-rest rotation about the second principal axis.

A homotopylike approach involving the parameter p was used to identify the switching structure for this nonlinear problem. For $p = 61$, none of the controls saturates and the final maneuver time is $t_f = 3.76$ s. For $p = 335$ (Fig. 2), both controls saturate during the maneuver and the final time is $t_f = 3.30$ s. Note that the control profile u_1 approaches a bang-bang-bang profile, whereas u_2 approaches a bang-off-bang profile. Thus, by sweeping the parameter p from lower to higher values in a homotopylike fashion, we have gained insight on the switching structure of the controls for the formal minimum-time solution of this example (see Ref. 10 for further details).

Closing Remarks

We have introduced a novel performance measure that gives rise to a one-parameter family of optimally smooth near-minimum-time controls. The optimal control formulation leads to a multipoint boundary-value problem, which may be solved using a multiple shooting method. This solution method appears to be computationally intense for systems in which the controls experience many switches; with all controls saturating, example 2 requires solving for 89 unknown parameters. We have found, however, that, by sweeping the parameter p in a homotopylike fashion, beginning with the condition that none of the controls saturates, insight to the optimal switching structure for the formal minimum-time problem may be gained. The problem formulation, numerical solution procedure, and example problems are covered in greater detail in Chap. 2 of Ref. 10.

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Time-Variant Receding-Horizon Control of Nonlinear Systems

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Introduction

THE objective of this Note is to extend the conventional real-time optimization algorithm[1] to the receding-horizon state feedback control problem with explicit time-variant parameters. An algorithm is derived in a modified manner, which results in less computation and less data storage than a direct extension of the conventional algorithm. A tracking control problem of a two-wheeled car is employed as a numerical example. A simulation result demonstrates closed-loop characteristics of the designed tracking control law.

Problem Formulation

The dynamic system treated here is expressed in the following differential equation:

$$\frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} = \boldsymbol{f}[\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}(t)] \tag{1}$$

where $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ denotes the state vector, $\boldsymbol{u}(t) \in \boldsymbol{R}^m$ the control input vector, and $\boldsymbol{p}(t) \in \boldsymbol{R}^r$ the vector of time-variant parameters. The time-variant parameter $\boldsymbol{p}(t)$ is assumed to be known. An optimal state feedback law is designed so as to minimize a receding-horizon performance index:

$$J = \varphi[\boldsymbol{x}(t+T), \boldsymbol{p}(t+T)] + \int_t^{t+T} L[\boldsymbol{x}(t'), \boldsymbol{u}(t'), \boldsymbol{p}(t')]\,\mathrm{d}t' \tag{2}$$

Because the time-variant parameter $\boldsymbol{p}(t)$ is included both in the state equation and in the performance index, the present problem can deal with not only control of time-variant systems but also tracking control problems. A command input is regarded as a time-variant parameter in a tracking control problem. The performance index evaluates the performance from the present time t to the finite future $t + T$. Because the time interval in the performance index is finite, the integrand in the performance index does not have to converge to zero as time increases. Therefore, the receding-horizon tracking control law can be determined even if the tracking error does not converge to zero with any control input. That is, any feasibility conditions on $\boldsymbol{p}(t)$ and any controllability conditions on the system are not necessary, although asymptotic tracking is not guaranteed, in general, by the present formulation of the receding-horizon tracking control. Some analytical results are found in Ref. 2 about the receding-horizon tracking control problem under restrictive conditions.

The performance index equation (2) is minimized for each time t starting from $\boldsymbol{x}(t)$. By denoting the trajectory $\boldsymbol{x}(t+\tau)$ starting from $\boldsymbol{x}(t)$ as $\boldsymbol{x}^*(\tau, t)$, the present receding-horizon control problem can be converted to a family of finite horizon optimal control problems on the τ axis parameterized by time t as follows.

Minimize:

$$J = \varphi[\boldsymbol{x}^*(T, t), \boldsymbol{p}(t+T)] + \int_0^T L[\boldsymbol{x}^*(\tau, t), \boldsymbol{u}^*(\tau, t), \boldsymbol{p}(t+\tau)]\,\mathrm{d}\tau \tag{3}$$

subject to:

$$\boldsymbol{x}^*_\tau(\tau, t) = \boldsymbol{f}[\boldsymbol{x}^*(\tau, t), \boldsymbol{u}^*(\tau, t), \boldsymbol{p}(t+\tau)] \tag{4}$$

with the initial state on the τ axis $\boldsymbol{x}^*(0, t)$ given by

$$\boldsymbol{x}^*(0, t) = \boldsymbol{x}(t) \tag{5}$$

The actual control input $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \boldsymbol{u}^*(0, t) \tag{6}$$

where the asterisk denotes a variable in the converted optimal control problem so as to distinguish it from its correspondence in the original problem. Note that the converted optimal control problem is a standard Bolza-type problem on the τ axis for a pair of fixed t and T and the actual control input is given by the initial value of the optimal control input minimizing the performance index. The first-order necessary conditions for the optimal solution are readily obtained as a two-point boundary-value problem (TPBVP) by the calculus of variations as[3]

$$\boldsymbol{x}^*_\tau(\tau, t) = H_\lambda^T, \qquad \boldsymbol{x}^*(0, t) = \boldsymbol{x}(t) \tag{7}$$

$$\boldsymbol{\lambda}^*_\tau(\tau, t) = -H_x^T, \qquad \boldsymbol{\lambda}^*(T, t) = \varphi_x^T[\boldsymbol{x}^*(T, t), \boldsymbol{p}(t+T)] \tag{8}$$

$$H_u = 0 \tag{9}$$

where $\boldsymbol{\lambda}^*(\tau, t) \in \boldsymbol{R}^n$ denotes the costate, H denotes the Hamiltonian defined as

$$H = L + \boldsymbol{\lambda}^{*T}\boldsymbol{f} \tag{10}$$

and H_x denotes the partial derivative of H with respect to $\boldsymbol{x}^*$, and so on.

Because the state and costate at $\tau = T$ are determined by the Euler–Lagrange equations (7–9) from the state and costate at $\tau = 0$, the TPBVP can be regarded as a nonlinear algebraic equation with respect to the costate at $\tau = 0$ as

$$\boldsymbol{F}[\boldsymbol{\lambda}(t), \boldsymbol{x}(t), T, t] = \boldsymbol{\lambda}^*(T, t) - \varphi_x^T[\boldsymbol{x}^*(T, t), \boldsymbol{p}(t+T)] = 0 \tag{11}$$

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